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The Dirac propagator in the extreme Kerr metric

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Abstract
Starting with the Dirac equation in the extreme Kerr metric we derive an integral representation for the propagator of solutions of the Cauchy problem with initial data in the class of smooth compactly supported functions.

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1. Introduction

One may think that the formation of an extreme Kerr black hole (EKBH) is of only academic interest but this is not so. Relativistic Dyson rings, i.e. uniformly rotating, homogeneous and axisymmetric relativistic fluid bodies with a toroidal shape admit a continuous transition to an EKBH if a fixed ratio $r_1/r_2 > 0.5613$ of inner to outer coordinate radius is prescribed and the gravitational mass gradually increases for fixed mass-density [1]. Concerning the existence of such rings based on numerical computations we refer to [2, 3]. Let us recall that relativistic Dyson rings could emerge from astrophysical scenarios like stellar core-collapses with high angular momentum [4] or they might simply be present in central regions of galaxies. Moreover, it has been proved that the only possible candidate for a black hole limit for stationary and axisymmetric, uniformly rotating perfect fluid bodies with a cold equation of state as well as for isentropic stellar models with a non-zero temperature is the EKBH [5]. Hence, we cannot *a priori* exclude that EKBHs play no role in astrophysics and in view of the above consideration it is of interest to study the Dirac equation in an extreme Kerr manifold.

It is surprising that there are no analytical studies concerning the propagation of Dirac fields outside an extreme Kerr black hole. With the present work we hope to fill this gap. Here, we prove for the first time the completeness of the Chandrasekhar ansatz for the Dirac equation in an EKBH from which an integral representation of the Dirac propagator can be obtained quite immediately.

An integral representation for the Dirac propagator in the non-extreme Kerr Newman metric has been derived in [6]. Although we use the same strategy as in [6] to write the Dirac equation in a Hamiltonian form the method used in our work to compute the Dirac propagator is more general and applies to a different situation. First, the propagator obtained in [6] has been derived under the conditions that the black hole is not extreme. In fact, in the extreme case you have to take into account the presence of bound states which are instead absent in the non-extreme case. Moreover, the method we used to derive the propagator is fundamentally different from that employed in [6]. In fact, we first show the completeness of the Chandrasekhar ansatz and from this we derive the integral representation for the Dirac propagator. The main advantage of our approach is that we do not need to consider the Hamiltonian operator in a finite box in order to construct the spectral measure as in [6]. Furthermore, the method used in [6] cannot be applied as it is without introducing some modifications.

The issue of the possibility of obtaining our results as a special limit of the non-extremal case is a very interesting one but it is not obvious at all how our result could be derived as a carefully extracted limit of the general case since the Dirac operators and the scalar products are defined on different Hilbert spaces. It is probably necessary to find an ad hoc notion of limit in a kind of parameterized family of Hilbert spaces in order to give a rigorous definition of the transition of the Dirac propagator from a non-extreme to an extreme black hole. Moreover, the ODEs arising from the Dirac equation after separation of variables in these two cases have a very different structure. Considering that the Cauchy horizon coincides with the event horizon in the extreme Kerr metric it would be interesting to find out how the confluence process from the radial Dirac equation in the non-extreme case to the extreme one depends on the black hole parameters. This approach might reveal some information to construct the transition from a non-extreme to an extreme black hole. We reserve the study of this limiting process for future investigations.

The rest of the paper is organized as follows. In section 2 we shortly derive the Dirac equation in the EKBH. After the introduction of the so-called Chandrasekhar ansatz we compute the scalar product with respect to which the Dirac Hamiltonian is formally self-adjoint. Section 3 is devoted to proving the completeness of the Chandrasekhar ansatz (see theorem 2.1) which in turn allows us to derive the integral representation for the Dirac propagator as given by (3.5).

2. The Dirac equation in the extreme Kerr metric

In Boyer–Lindquist coordinates $(t, r, \vartheta, \varphi)$ with $r > 0$, $0 \leqslant \vartheta \leqslant \pi$, $0 \leqslant \varphi < 2\pi$ the extreme Kerr metric can be easily derived from the Kerr metric [7] by setting the Kerr parameter $a = M$. Its form is given by

$$\mathrm{d}s^2 = \left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{4M^2 r\sin^2\vartheta}{\Sigma}\,\mathrm{d}t\,\mathrm{d}\varphi - \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 - \Sigma\,\mathrm{d}\vartheta^2 - (r^2+M^2)^2\sin^2\vartheta\,\frac{\widetilde{\Sigma}}{\Sigma}\,\mathrm{d}\varphi^2 \tag{2.1}$$

with

$$\Sigma := \Sigma(r,\vartheta) = r^2 + M^2\cos^2\vartheta, \qquad \Delta := \Delta(r) = (r-M)^2$$

and

$$\widetilde{\Sigma} := \widetilde{\Sigma}(r,\vartheta) = 1 - M^2\gamma^2(r)\sin^2\vartheta, \qquad \gamma(r) := \frac{r-M}{r^2+M^2}$$

where M is the mass of a spinning black hole with angular momentum $J = M^2$. Note that the area of an EKBH is simply $A = 8\pi J$. Since the equation $\Delta = 0$ has a double root at $r_0 := M$ the Cauchy horizon and the event horizon coincide. Finally, note that $\widetilde{\Sigma} > 0$ for all $r > M$ and $\vartheta \in [0, \pi]$.

According to Penrose and Rindler [8] the Dirac equation coupled to a general gravitational field $\mathbf{V}$ is given in terms of two-component spinors $(\phi^A, \chi^{A'})$ by

$$(\nabla^A_{A'} - \mathrm{i}eV^A_{A'})\phi_A = \frac{m_e}{\sqrt{2}}\chi_{A'}, \qquad (\nabla^{A'}_A - \mathrm{i}eV^{A'}_A)\chi_{A'} = \frac{m_e}{\sqrt{2}}\phi_A$$

where we used Planck units $\hbar = c = G = 1$. Furthermore, $\nabla_{AA'}$ is the symbol for covariant differentiation, e is the charge or coupling constant of the Dirac particle to the vector field $\mathbf{V}$ and m_e is the particle mass. The Dirac equation in the Kerr geometry was computed and separated by Chandrasekhar [9] with the help of the Kinnersley tetrad [10]. Since the derivation and separation of the Dirac equation in the extreme Kerr geometry follows with minor changes from section 2 in [11] we will limit us to give here only the main results. In view of the separation of the Dirac equation we choose to work with the Carter tetrad [12]. A Dirac spinor $\psi = \psi(t, r, \vartheta, \varphi) \in \mathbb{C}^4$ satisfies in the exterior region $r \in (M, \infty)$ of an extreme Kerr black hole the following equation:

$$\mathcal{W}\psi = \left(\mathcal{W}_{(t,r,\varphi)} + \mathcal{W}_{(t,\vartheta,\varphi)}\right)\psi = 0 \tag{2.2}$$

where

$$\mathcal{W}_{(t,r,\varphi)} = \begin{pmatrix} \mathrm{i}m_e r & 0 & \sqrt{\Delta}\mathcal{D}_+ & 0 \\ 0 & -\mathrm{i}m_e r & 0 & \sqrt{\Delta}\mathcal{D}_- \\ \sqrt{\Delta}\mathcal{D}_- & 0 & -\mathrm{i}m_e r & 0 \\ 0 & \sqrt{\Delta}\mathcal{D}_+ & 0 & \mathrm{i}m_e r \end{pmatrix}, \tag{2.3}$$

$$\mathcal{W}_{(t,\vartheta,\varphi)} = \begin{pmatrix} -Mm_e\cos\vartheta & 0 & 0 & \mathcal{L}_+ \\ 0 & Mm_e\cos\vartheta & -\mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ & -Mm_e\cos\vartheta & 0 \\ -\mathcal{L}_- & 0 & 0 & Mm_e\cos\vartheta \end{pmatrix} \tag{2.4}$$

with $\mathcal{D}_\pm$ and $\mathcal{L}_\pm$ defined by

$$\mathcal{D}_\pm = \frac{\partial}{\partial r} \mp \frac{1}{\Delta}\left[(r^2 + M^2)\frac{\partial}{\partial t} + M\frac{\partial}{\partial\varphi}\right], \tag{2.5}$$

$$\mathcal{L}_\pm = \frac{\partial}{\partial\vartheta} + \frac{1}{2}\cot\vartheta \mp \mathrm{i}\left(M\sin\vartheta\frac{\partial}{\partial t} + \csc\vartheta\frac{\partial}{\partial\varphi}\right). \tag{2.6}$$

By means of the Chandrasekhar ansatz [9, 11]

$$\psi(t, r, \vartheta, \varphi) = \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{\mathrm{i}\kappa\varphi}\begin{pmatrix} R_-(r)S_-(\vartheta) \\ R_+(r)S_+(\vartheta) \\ R_+(r)S_-(\vartheta) \\ R_-(r)S_+(\vartheta) \end{pmatrix}, \qquad \kappa := k + \frac{1}{2}$$

where ω and $k \in \mathbb{Z}$ denote the energy of the particle and its azimuthal quantum number respectively, the Dirac equation decouples into the following systems of linear first-order differential equations for the radial $R_\pm$ and angular components $S_\pm$ of the spinor ψ,

$$\begin{pmatrix} \sqrt{\Delta}\widehat{\mathcal{D}}_- & -\mathrm{i}m_e r - \lambda \\ \mathrm{i}m_e r - \lambda & \sqrt{\Delta}\widehat{\mathcal{D}}_+ \end{pmatrix}\begin{pmatrix} R_- \\ R_+ \end{pmatrix} = 0, \tag{2.7}$$

$$\begin{pmatrix} -\widehat{\mathcal{L}}_- & \lambda + Mm_e\cos\theta \\ \lambda - Mm_e\cos\theta & \widehat{\mathcal{L}}_+ \end{pmatrix}\begin{pmatrix} S_- \\ S_+ \end{pmatrix} = 0 \tag{2.8}$$

where

$$\widehat{\mathcal{D}}_{\pm} = \frac{\mathrm{d}}{\mathrm{d}r} \mp \mathrm{i}\frac{K(r)}{\Delta}, \qquad K(r) = \omega(r^2 + M^2) + \kappa M, \tag{2.9}$$

$$\widehat{\mathcal{L}}_{\pm} = \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{1}{2}\cot\vartheta \pm Q(\vartheta), \qquad Q(\vartheta) = M\omega\sin\vartheta + \kappa\csc\vartheta \tag{2.10}$$

and λ is a separation constant depending on k and ω. Let $u \in \mathbb{R}$ be the tortoise coordinate defined by $\mathrm{d}u/\mathrm{d}r = (r^2 + M^2)/\Delta$. By rearranging (2.2) we can write the Dirac equation in the Hamiltonian form

$$\mathrm{i}\frac{\partial\psi}{\partial t} = H\psi, \qquad H = H_0 + V(u, \vartheta) \tag{2.11}$$

with

$$H_0 = A(u,\vartheta)\left[\begin{pmatrix} -\mathcal{E}_- & 0 & 0 & 0 \\ 0 & \mathcal{E}_+ & 0 & 0 \\ 0 & 0 & \mathcal{E}_+ & 0 \\ 0 & 0 & 0 & -\mathcal{E}_- \end{pmatrix} + \begin{pmatrix} 0 & -\mathcal{M}_+ & 0 & 0 \\ -\mathcal{M}_- & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{M}_+ \\ 0 & 0 & \mathcal{M}_- & 0 \end{pmatrix}\right], \tag{2.12}$$

and

$$A(u,\vartheta) = \frac{1}{\widetilde{\Sigma}}\left[\mathbb{1}_4 - M\gamma(u)\sin\vartheta\begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}\right], \tag{2.13}$$

$$V(u,\vartheta) = m_e A(u,\vartheta)\gamma(u)\begin{pmatrix} 0 & \frac{\mathbb{1}_2}{\widetilde{\rho}} \\ \frac{\mathbb{1}_2}{\overline{\widetilde{\rho}}} & 0 \end{pmatrix} \tag{2.14}$$

where σ_2 is a Pauli matrix, $\widetilde{\rho} = -(r - \mathrm{i}M\cos\vartheta)^{-1}$ and

$$\mathcal{E}_{\pm} = \mathrm{i}\left(\frac{\partial}{\partial u} \mp \frac{M}{r^2 + M^2}\frac{\partial}{\partial\varphi}\right), \qquad \mathcal{M}_{\pm} = \mathrm{i}\gamma(u)\left(\frac{\partial}{\partial\vartheta} + \frac{1}{2}\cot\vartheta \mp \mathrm{i}\csc\vartheta\frac{\partial}{\partial\varphi}\right)$$

satisfying $\overline{\mathcal{E}}_{\pm} = -\mathcal{E}_{\pm}$ and $\overline{\mathcal{M}}_{\pm} = -\mathcal{M}_{\mp}$. Notice that the matrix contained in the square brackets in (2.13) is hermitian. Similarly as in section 3 in [11] we can construct a positive scalar product

$$\langle\psi|\phi\rangle = \int_{-\infty}^{+\infty}\mathrm{d}u\int_{-1}^{1}\mathrm{d}(\cos\vartheta)\int_0^{2\pi}\mathrm{d}\varphi\sqrt{\widetilde{\Sigma}}\,\overline{\psi}\phi \tag{2.15}$$

with respect to which the Hamiltonian H acting on the spinor ψ on the hypersurface $t = t_0$ with t_0 constant is formally self-adjoint. In the following we consider the Hilbert space $\mathcal{H} = L_2(\Omega)^4 := L_2(\Omega, \sqrt{\widetilde{\Sigma}}\ \mathrm{d}u\,\mathrm{d}(\cos\vartheta)\,\mathrm{d}\varphi)^4$ consisting of wavefunctions $\psi : \Omega := \mathbb{R}\times S^2 \to \mathbb{C}^4$ together with the positive scalar product (2.15). As in [11], it can be shown that the Hamiltonian H defined on $\mathcal{C}_0^{\infty}(\Omega)^4$ is essentially self-adjoint and has a unique self-adjoint extension. System (2.8) can be brought in the so-called Dirac form

$$(\mathcal{U}S)(\vartheta) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{\mathrm{d}S}{\mathrm{d}\vartheta} + \begin{pmatrix} -Mm_e\cos\vartheta & -\frac{\kappa}{\sin\vartheta} - M\omega\sin\vartheta \\ -\frac{\kappa}{\sin\vartheta} - M\omega\sin\vartheta & Mm_e\cos\vartheta \end{pmatrix} S = \lambda S$$

with $S(\vartheta) = \sqrt{\sin\vartheta}(S_-(\vartheta), S_+(\vartheta))^T$ and $\vartheta \in (0, \pi)$. In $L_2(0,\pi)^2$ the angular operator $\mathcal{U}$ with domain $D(\mathcal{U}) = \mathcal{C}_0^{\infty}(0,\pi)^2$ is essentially self-adjoint, its spectrum is discrete and consists of simple eigenvalues, i.e. $\lambda_j < \lambda_{j+1}$ for every $j \in \mathbb{Z}\backslash\{0\}$. Moreover, the eigenvalues depend smoothly on ω. Furthermore, the functions $\mathrm{e}^{\mathrm{i}(k+\frac{1}{2})\varphi}$ are eigenfunctions of the z-component of the total angular momentum operator $\widehat{J}_z$ with eigenvalues $-\left(k+\frac{1}{2}\right)$ with $k \in \mathbb{Z}$. Since energy, generalized squared total angular momentum and the z-component of the total angular momentum form a set of commuting observables $\{H, \widehat{J}^2, \widehat{J}_z\}$ we will label the generalized states $\psi \in \mathcal{H}$ by ψ_ω^{kj}. Finally, for every $k \in \mathbb{Z}$ and $j \in \mathbb{Z}\backslash\{0\}$ the set $\left\{Y_\omega^{kj}(\vartheta, \varphi)\right\}$ with

$$Y_\omega^{kj}(\vartheta,\varphi) = \begin{pmatrix} Y_{\omega,-}^{kj}(\vartheta,\varphi) \\ Y_{\omega,+}^{kj}(\vartheta,\varphi) \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} S_{\omega,-}^{kj}(\vartheta) \\ S_{\omega,+}^{kj}(\vartheta) \end{pmatrix} \mathrm{e}^{\mathrm{i}\left(k+\frac{1}{2}\right)\varphi} \tag{2.16}$$

is a complete orthonormal basis for $L_2(S^2)^2$ [11]. Let $\sigma(H) \subseteq \mathbb{R}$ denote the spectrum of the self-adjoint Hamiltonian operator H. We state now the theorem on the completeness of the Chandrasekhar ansatz.

Theorem 2.1. *For every* $\psi \in \mathcal{C}_0^\infty(\Omega)^4$

$$\psi(0,x) = \int_{\sigma(H)} \sum_{j\in\mathbb{Z}\backslash\{0\}} \sum_{k\in\mathbb{Z}} \left\langle \psi_\omega^{kj} \middle| \psi_\omega \right\rangle \psi_\omega^{kj}(x)\, \mathrm{d}\mu_\omega, \quad \psi_\omega^{kj}(x) = \begin{pmatrix} R_{\omega,-}^{kj}(u) Y_{\omega,-}^{kj}(\vartheta,\varphi) \\ R_{\omega,+}^{kj}(u) Y_{\omega,+}^{kj}(\vartheta,\varphi) \\ R_{\omega,+}^{kj}(u) Y_{\omega,-}^{kj}(\vartheta,\varphi) \\ R_{\omega,-}^{kj}(u) Y_{\omega,+}^{kj}(\vartheta,\varphi) \end{pmatrix} \tag{2.17}$$

where the scalar product $\langle\cdot|\cdot\rangle$ *is given by (2.15),* μ_ω *is a Borel measure on* $\sigma(H) \subseteq \mathbb{R}$ *and* $x = (u, \vartheta, \varphi)$.

Proof. We show first that it is possible to construct isometric operators

$$\widehat{W}_{k,j} : \mathcal{C}_0^\infty(\mathbb{R})^2 \longrightarrow \mathcal{C}_0^\infty(\Omega)^4,$$

such that

$$R_\omega^{kj}(u) = \begin{pmatrix} R_{\omega,-}^{kj}(u) \\ R_{\omega,+}^{kj}(u) \end{pmatrix} \longmapsto \mathcal{A}(u,\vartheta) \begin{pmatrix} R_{\omega,-}^{kj}(u) Y_{\omega,-}^{kj}(\vartheta,\varphi) \\ R_{\omega,+}^{kj}(u) Y_{\omega,+}^{kj}(\vartheta,\varphi) \\ R_{\omega,+}^{kj}(u) Y_{\omega,-}^{kj}(\vartheta,\varphi) \\ R_{\omega,-}^{kj}(u) Y_{\omega,+}^{kj}(\vartheta,\varphi) \end{pmatrix}$$

with some function $\mathcal{A}$ to be determined and $Y_{\omega,\pm}^{kj}$ given by (2.16). Indeed, since the angular eigenfunctions Y_ω^{kj} are normalized, we have

$$\left\| R_\omega^{kj} \right\|_{L_2(\mathbb{R})^2}^2 = \int_{-\infty}^{+\infty} \mathrm{d}u \int_{-1}^{1} d(\cos\vartheta) \int_0^{2\pi} \mathrm{d}\varphi \overline{\psi}_\omega^{kj} \psi_\omega^{kj}, \qquad \psi_\omega^{kj} = \begin{pmatrix} R_{\omega,-}^{kj} Y_{\omega,-}^{kj} \\ R_{\omega,+}^{kj} Y_{\omega,+}^{kj} \\ R_{\omega,+}^{kj} Y_{\omega,-}^{kj} \\ R_{\omega,-}^{kj} Y_{\omega,+}^{kj} \end{pmatrix}$$

and by choosing $\mathcal{A} = (\widetilde{\Sigma})^{-\frac{1}{4}}$ it results that $\left\| R_\omega^{kj} \right\|_{L_2(\mathbb{R})^2}^2 = \left\| \widehat{W}_{k,j}\left(R_\omega^{kj}\right) \right\|_{L_2(\Omega)^4}^2$. By means of the isometric operators $\widehat{W}_{k,j}$ we can now introduce for every $\omega \in \sigma(H)$ an auxiliary separable Hilbert space $\mathfrak{h}(\omega)$ as follows:

$$\mathfrak{h}(\omega) = \bigoplus_{j\in\mathbb{Z}\backslash 0} \bigoplus_{k\in\mathbb{Z}} \mathfrak{h}_{k,j}, \qquad \mathfrak{h}_{k,j} = \widehat{W}_{k,j}\left(\mathcal{C}_0^\infty(\mathbb{R})^2\right).$$

According to the expansion theorem (e.g. theorem 3.7 in [14]) every element ψ_ω in $\mathfrak{h}(\omega)$ can be written as

$$\psi_\omega = \sum_{j\in\mathbb{Z}\backslash 0}\sum_{k\in\mathbb{Z}} \langle \psi_\omega^{kj} | \psi_\omega \rangle \psi_\omega^{kj}. \tag{2.18}$$

Finally, the direct integral of Hilbert spaces

$$\mathfrak{H} = \int_{\sigma(H)} \bigoplus_{j\in\mathbb{Z}\backslash 0} \bigoplus_{k\in\mathbb{Z}} \mathfrak{h}_{k,j}\, \mathrm{d}\mu_\omega \tag{2.19}$$

with μ_ω a Borel measure on $\sigma(H) \subseteq \mathbb{R}$ is defined (see ch. 1, section 5.1 in [15]) as the Hilbert space of vector-valued functions ψ_ω taking values in the auxiliary Hilbert spaces $\mathfrak{h}(\omega)$. By definition $\mathcal{H}$ can be written as in (2.19) if there exists a unitary mapping $\mathcal{F}$ of $\mathcal{H}$ onto $\mathfrak{H}$. Now, since the Hamiltonian H is self-adjoint, the spectral representation theorem (e.g. theorem 7.18 in [14]) implies the existence of such an isomorphism $\mathcal{F}$ and this completes the proof. □

At this point note that in general for self-adjoint operators $\sigma(H) = \sigma_p(H) \cup \sigma_c(H)$ where $\sigma_p(H)$ and $\sigma_c(H)$ denote the point spectrum and the continuous spectrum, respectively [16]. Furthermore, we distinguish between points of $\sigma_p(H)$ which are isolated or non-isolated as points of $\sigma(H)$. The former constitutes the discrete spectrum $\sigma_d(H)$ which is defined as the subset of $\sigma(H)$ for which the resolvent is closed. The latter is called the point continuous spectrum $\sigma_{pc}(H)$. In the next section we investigate the spectrum of the Hamiltonian H in order to compute the measure μ_ω entering in (2.17).

3. The Dirac propagator

According to theorem 2.1 every representative element $\psi_\omega = (\mathcal{F}\psi)(\omega)$ of the element $\psi \in \mathcal{H}$ in the decomposition (2.19) can be written in the form given by (2.18). Hence, in the study of $\sigma(H)$ we are allowed to focus our analysis on the radial system (2.7). As a consequence, we just need to investigate the spectrum of the differential operator $\mathcal{R}$ associated with the formal differential system (2.7) after it is brought into the form of a Dirac system of ordinary differential equations. This can be achieved in two steps. First, we set $R_-(r) = F(r) + \mathrm{i}G(r)$ and $R_+(r) = F(r) - \mathrm{i}G(r)$ in (2.7) and obtain the following system:

$$\frac{\mathrm{d}F}{\mathrm{d}r} = \frac{\lambda}{r-M}F + \left[\frac{K(r)}{(r-M)^2} + \frac{m_e r}{r-M}\right]G,$$
$$\frac{\mathrm{d}G}{\mathrm{d}r} = -\frac{\lambda}{r-M}G + \left[-\frac{K(r)}{(r-M)^2} + \frac{m_e r}{r-M}\right]F.$$

By introducing the tortoise coordinate $u \in \mathbb{R}$ defined in section 2 the above equations give the following first-order system, namely

$$\frac{\mathrm{d}\Xi}{\mathrm{d}u} = \begin{pmatrix} \lambda\Delta\Omega & \omega + \kappa\Omega + m_e r(u)\Delta\Omega \\ -\omega - \kappa\Omega + m_e r(u)\Delta\Omega & -\lambda\Delta\Omega \end{pmatrix}\Xi \tag{3.1}$$

with $\Xi := (F, G)^T$ and

$$\Omega := \Omega(u) = \frac{M}{r(u)^2 + M^2}, \qquad \Delta\Omega := \Delta\Omega(u) := \frac{r(u)}{r^2(u) + M^2} - \Omega(u).$$

For ease in notation we shall write r instead of $r(u)$ when no risk of confusion arises. Finally, we rewrite (3.1) as follows:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\mathrm{d}\Xi}{\mathrm{d}u} + \begin{pmatrix} -\kappa\Omega - m_e r\Delta\Omega & -\lambda\Delta\Omega \\ -\lambda\Delta\Omega & -\kappa\Omega + m_e r\Delta\Omega \end{pmatrix}\Xi = \omega\Xi. \tag{3.2}$$

Notice that the value of Ω at the event horizon is simply the black hole angular velocity $\Omega_H = 1/(2M)$. Moreover, $\Delta\Omega \to 0$ for $u \to -\infty$. Let us now consider the formal differential operator

$$\tau := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}u} - \begin{pmatrix} \kappa\Omega + m_e r \Delta\Omega & \lambda\Delta\Omega \\ \lambda\Delta\Omega & \kappa\Omega - m_e r \Delta\Omega \end{pmatrix} \tag{3.3}$$

in the Hilbert space $L_2(\mathbb{R}, \mathrm{d}u)^2$. Theorem 6.8 in [17] implies that τ is in the limit point case (l.p.c.) at $\pm\infty$. Hence, the deficiency indices (γ_+, γ_-) of τ are (0, 0), its deficiency numbers are $(\gamma_{-\infty}, \gamma_\infty) = (1, 1)$ and τ admits only one self-adjoint extension T with $D(T) = \mathcal{C}_0^\infty(\mathbb{R})^2$ (see ch. 4, p 53 and theorem 5.7-8 in [17]).

Theorem 3.1. *Let T with $D(T) = \mathcal{C}_0^\infty(\mathbb{R})^2$ be the self-adjoint extension of the formal differential operator τ defined in (3.3). Then, $\sigma_e(T) = \mathbb{R} = \sigma(T)$ where $\sigma_e(\cdot)$ denotes the essential spectrum.*

Proof. In order to determine the essential spectrum of T we apply the so-called decomposition method due to Neumark [18]. For this purpose let T_- and T_+ be self-adjoint extensions of the operator T restricted to the intervals $(-\infty, 0]$ and $[0, \infty)$, respectively. Moreover, let the 2×2 symmetric matrix $P(u)$ be defined as follows:

$$P(u) := \begin{pmatrix} -\kappa\Omega - m_e r \Delta\Omega & -\lambda\Delta\Omega \\ -\lambda\Delta\Omega & -\kappa\Omega + m_e r \Delta\Omega \end{pmatrix}.$$

A straightforward computation shows that

$$P_0 := \lim_{u\to+\infty} P(u) = \begin{pmatrix} -m_e & 0 \\ 0 & m_e \end{pmatrix}, \qquad P_1 := \lim_{u\to-\infty} P(u) = \begin{pmatrix} -\kappa\Omega_H & 0 \\ 0 & -\kappa\Omega_H \end{pmatrix}.$$

Let $\mu_\pm^{(i)}$ with $i = 0, 1$ denote the eigenvalues of P_0 and P_1, respectively. Since $\mu_-^{(0)} = -m_e$, $\mu_+^{(0)} = m_e$ and $\mu_-^{(1)} = \mu_+^{(1)} = -\kappa\Omega_H = -\kappa/(2M)$ theorem 16.5 in [17] implies that

$$\sigma_e(T_+) \cap (-m_e, m_e) = \emptyset, \qquad \sigma_e(T_-) \cap \emptyset = \emptyset.$$

Notice that the applicability of theorem 16.5 continues to hold also for the case $m_e = 0$. Let $d \in (0, \infty)$. A simple computation shows that for $u \to +\infty$

$$P(u) - P_0 = \begin{pmatrix} m_e M & -\lambda \\ -\lambda & -m_e M \end{pmatrix} \frac{1}{u} + \mathcal{O}\left(\frac{1}{u^2}\right).$$

Hence, it follows that

$$\lim_{x\to+\infty} \frac{1}{x} \int_d^x \mathrm{d}u |P(u) - P_0| = 0$$

and theorem 16.6 in [17] implies that

$$\sigma_e(T_+) \supset (-\infty, m_e] \cup [m_e, \infty).$$

Let $\delta \in (-\infty, 0)$. In the limit $r \to M$ we have

$$P(r) - P_1 = \frac{1}{M} \begin{pmatrix} \Omega_H(\kappa - m_e M) & -\lambda/(2M) \\ -\lambda/(2M) & \Omega_H(\kappa + m_e M) \end{pmatrix} (r - M) + \mathcal{O}((r - M)^2)$$

and

$$\frac{\mathrm{d}u}{\mathrm{d}r} = 1 + \frac{2M}{r - M} + \frac{2M^2}{(r - M)^2}, \qquad \frac{1}{u} = -\frac{1}{2M^2}(r - M) + \mathcal{O}((r - M)^2).$$

Now, it can be easily checked that

$$\lim_{y\to-\infty}\frac{1}{y}\int_y^{\delta}\mathrm{d}u|P(u)-P_1| = \lim_{r\to M}\frac{1}{y(r)}\int_r^{\widetilde{\delta}}\mathrm{d}\widetilde{r}\frac{\mathrm{d}u}{\mathrm{d}\widetilde{r}}|P(\widetilde{r})-P_1| = 0$$

and theorem 16.6 implies that $\sigma_e(T_-)\supset\mathbb{R}$. Finally, the remark to theorem 11.5 in [17] yields that

$$\sigma_e(T) = \sigma_e(T_-)\cup\sigma_e(T_+) = \mathbb{R}. \qquad \square$$

From theorem 3.1 it results that $\sigma_e(H) = \mathbb{R} = \sigma(H)$. Since for the discrete spectrum $\sigma_d(H) = \sigma(H)\backslash\sigma_e(H)$ we find that $\sigma_d(H) = \emptyset$. Hence, the absolutely continuous spectrum of H is simply $\sigma_{ac}(H) = \mathbb{R}$ and the purely point spectrum is empty, i.e. $\sigma_{pp}(H) = \emptyset$. On the other hand the Dirac Hamiltonian in the extreme Kerr metric admits an eigenvalue $\Omega_k = -\kappa\Omega_H$ for $|\omega| < m_e$ and for each $\kappa = k+1/2$ with $k\in\mathbb{Z}$ [19]. Hence, the point continuous spectrum $\sigma_{pc}(H)$ is not empty since $\sigma_{pc}(H) = \{\Omega_k\}$ for k fixed. Thus we can conclude that such an eigenvalue is embedded in the continuous spectrum of H. Moreover, the Lebesgue decomposition theorem leads to a decomposition of the spectral measure μ into the sum of a part absolutely continuous with respect to Lebesgue measure and a singular part, i.e. $\mu = \mu_{ac}+\mu_s$ and the Radon–Nikodym theorem implies that the absolutely continuous part $\mu_{ac,\omega}$ of the spectral measure may be described by

$$\mu_{ac,\omega} = \int_{\mathbb{R}}\mathrm{d}\omega f(\omega)$$

with $f(\omega)$ a density function defined as $f(\omega) = \mathrm{d}\rho(\omega)/\mathrm{d}\omega$ for almost all $\omega\in\mathbb{R}$ and $\rho(\omega)$ the spectral function. Since the spectral measure of the operator H coincides with the Lebesgue measure on the spectrum of H (see theorem 3.1, p 447 and ch. VI, section 5 in [20]) the singular component $\mu_{s,\omega}$ is supported on the set of eigenvalues of the operator H. These may be characterized as the points of discontinuity of the spectral function $\rho(\omega)$, i.e. points where $\rho(\omega)$ has an isolated jump. Without loss of generality we can choose $\mu_{s,\omega} = \widetilde{H}(\omega-\Omega_k)$ where $\widetilde{H}$ is the Heaviside function. Taking into account that in the distributional sense the derivative of the Heaviside function gives rise to a Dirac-delta, we find that (2.17) becomes

$$\psi(x) = \int_{\mathbb{R}}\mathrm{d}\omega\sum_{j\in\mathbb{Z}\backslash\{0\}}\sum_{k\in\mathbb{Z}}\langle\psi_\omega^{kj}|\psi_\omega\rangle\psi_\omega^{kj}(x) + \sum_{j\in\mathbb{Z}\backslash\{0\}}\sum_{k\in\mathbb{Z}}\langle\psi_{\Omega_k}^{kj}|\psi_{\Omega_k}\rangle\psi_{\Omega_k}^{kj}(x) \tag{3.4}$$

with

$$\psi_\omega^{kj}(x) = \begin{pmatrix} R_{\omega,-}^{kj}(u)Y_{\omega,-}^{kj}(\vartheta,\varphi)\\ R_{\omega,+}^{kj}(u)Y_{\omega,+}^{kj}(\vartheta,\varphi)\\ R_{\omega,+}^{kj}(u)Y_{\omega,-}^{kj}(\vartheta,\varphi)\\ R_{\omega,-}^{kj}(u)Y_{\omega,+}^{kj}(\vartheta,\varphi)\end{pmatrix}, \qquad \psi_{\Omega_k}^{kj}(x) = \begin{pmatrix} R_{\Omega_k,-}^{kj}(u)Y_{\Omega_k,-}^{kj}(\vartheta,\varphi)\\ R_{\Omega_k,+}^{kj}(u)Y_{\Omega_k,+}^{kj}(\vartheta,\varphi)\\ R_{\omega_k,+}^{kj}(u)Y_{\omega_k,-}^{kj}(\vartheta,\varphi)\\ R_{\omega_k,-}^{kj}(u)Y_{\omega_k,+}^{kj}(\vartheta,\varphi)\end{pmatrix}$$

where $R_{\Omega_k,\pm}^{kj}$ and $Y_{\Omega_k,\pm}^{kj}$ are the radial and angular eigenfunctions satisfying (2.7) and (2.8) with $\omega = \Omega_k$, respectively. For more details on such eigenfunctions we refer to theorem 3.6 in [19]. Finally, since the Hamiltonian H is self-adjoint the spectral theorem implies that for every $\psi\in\mathcal{C}_0^\infty(\Omega)^4$

$$\psi(t,x) = \int_{\mathbb{R}}\mathrm{d}\omega\,\mathrm{e}^{\mathrm{i}\omega t}\sum_{j\in\mathbb{Z}\backslash\{0\}}\sum_{k\in\mathbb{Z}}\langle\psi_\omega^{kj}|\psi_\omega\rangle\psi_\omega^{kj}(x) + \sum_{j\in\mathbb{Z}\backslash\{0\}}\sum_{k\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}\Omega_k t}\langle\psi_{\Omega_k}^{kj}|\psi_{\Omega_k}\rangle\psi_{\Omega_k}^{kj}(x). \tag{3.5}$$

The above expression is the integral representation of the Dirac propagator in an extreme Kerr manifold.

4. Conclusions

In this paper we have derived an integral representation for the propagator of a massive fermion in the extreme Kerr geometry. Such a representation followed quite immediately by proving the completeness of the Chandrasekhar ansatz for the Dirac equation in an extreme Kerr manifold. In the future starting with a refined analysis of the Dirac propagator (3.5) we reserve to derive precise formulae for the probabilities that a massive Dirac particle disappears into the extreme Kerr black hole, forms a bound state with it or escapes at infinity for smooth initial data with compact support outside the event horizon and bounded angular momentum. Formulae for probabilities concerning the evolution of a Dirac particle have been derived in the non-extreme case in [21].

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